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History of Applied Mathematics

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From Abel to Somov: Elliptic Functions in Solving Rigid Body Rotation Problems

Abstract. The work is devoted to the application of the concept of an elliptical function to mechanics problems, mainly to solve the problem of rotation of rigid bodies. We will focus our attention on the most significant results in this regard contained in the works of Abel, Jacobi, Weierstrass and Somov. Starting from the proof of Abel's theorem, the representation of elliptic functions in terms of theta functions is shown. The article contains a sketch of a solution to the classical problem of rotation of a rigid body around a fixed point using hypereliptic functions.

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Key words and phrases: Elliptic function, Jacobi, Abel, Somov.

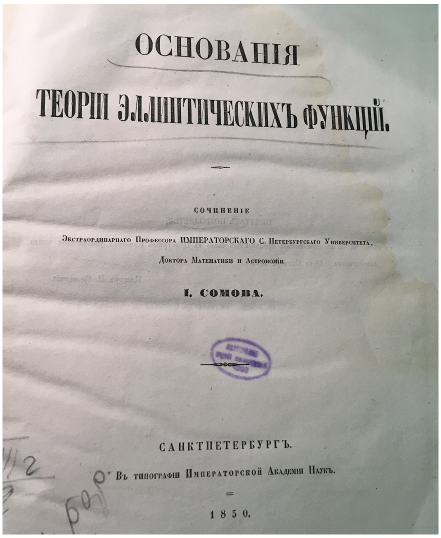
1. Introduction. The use of elliptic and hyperelliptic functions reduces the problem of the rotation of a rigid body around a fixed point to the simplest elements. The sought parameters of motion (direction cosines of the Euler angles) are the composition of such functions.

The first systematic presentation of the theory of elliptic functions in Russia was made by the St. Petersburg academician Osip Ivanovich Somov. This branch of integral calculus, and still not easy, was stated by Somov clearly and in detail in his fundamental work "Foundations of the theory of elliptic functions" (1850). The book contains seven chapters, and another additional chapter is devoted to the applications of elliptic functions to some problems of geometry and mechanics. By the middle of the nineteenth century, the following apparatus was formed in the theory of elliptic functions.

2. Mathematical preamble. The concept of an elliptic function appeared in the mid-17th century (e.g. the elliptic integral in Wallis (1655); Jacobi's opinion the 23rd of December 1751 is "the birthday

[1]Осип (Иосиф) Иванович Сомов From Digital Collection (2022)

Figure 1: Osip Ivanovich Somov (1815-1876)[1].



ОСНОВАНІЯ

ТЕОРІИ ЭЛЛИПТИЧЕСКИХЪ ФУНКЦІЙ.

СОЧИНЕНІЕ

Экстраординарнаго Профессора ИМПЕРАТОРСКАГО С. Петербургскаго Университета,

Доктора Математики и Астрономіи

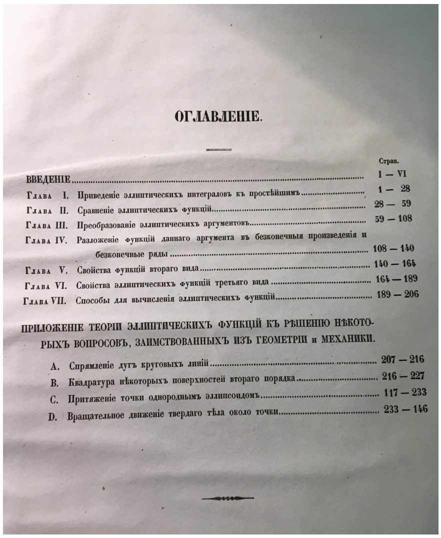
I. СОМОВА.

САНКТПЕТЕРБУРГЪ.

Въ типографіи Императорской Академіи Наукъ.

1850.

(a) The title page.



ОГЛАВЛЕНІЕ.

(b) Contents

Figure 2: "Foundations of the theory of elliptic functions" by Сомов, Осип Иванович published in 1850.

of elliptic functions" v. [3, p. 183], [16, p.23]). In order to present the applications of this class of functions in applications in mechanics, we will show the reduction, comparison, and transformation of such functions by Somov (1850), Psheborskiy (1895, Tikhomandritskiy (1885). When the integral of an algebraic function is not reduced to another algebraic function, it is considered a transcendental function. These are integrals of the form $z = \int f(x, \sqrt{R})dx$, where R is an entire function with respect to x, and f is rational with respect to x and R. If R has the first or second degree, then the corresponding integral $z = \int f(x, \sqrt{R})dx$ can be reduced either to an algebraic function or to an inverse trigonometric function (circular functions). If the degree is third or higher, then such integrals are called elliptic, and the inversion of such integrals, respectively, are called elliptic functions. The calculating problem the arc length of an ellipse gave the name to these functions.

The equation defining the relationship between algebraic functions and integrals of the form $\int \frac{x^m dx}{\sqrt{R}}$ is written as follows:
$b_0 \int \frac{dx}{\sqrt{R}} + b_1 \int \frac{xdx}{\sqrt{R}} + b_2 \int \frac{x^2 dx}{\sqrt{R}} + \ldots + b_m \int \frac{x^m dx}{\sqrt{R}} = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-3} x^{m-3})\sqrt{R}$.
This equation makes it possible to replace the elliptic integral with the simplest algebraic expansions in the case when $m > 2$.

The factors at $\sqrt{R}$ on the right side of the equation must contain only positive powers of x (initial assumption), then the number m must be at least 3. An important consequence is obtained from this: in the case when the degree of an entire function P is less than 3, the integral $\int \frac{Pdx}{\sqrt{R}}$ cannot be expressed by an algebraic function. Therefore

$$\int \frac{dx}{\sqrt{R}}, \int \frac{xdx}{\sqrt{R}}, \int \frac{x^2 dx}{\sqrt{R}}$$

have no algebraic representation and are transcendental with respect to the function x.

Similar conclusions are valid in the case when P is a regular fraction $P = \frac{A}{(x-\alpha)^m}$, where A and α are constants, and m is a positive integer. Then, the considered integral $\int \frac{Pdx}{\sqrt{R}}$ decomposes into terms of the form

$$A \int \frac{dx}{(x-\alpha)^m \sqrt{R}}$$

The presented integral can be reduced to the simplest ones in the same way as $\int \frac{x^m dx}{\sqrt{R}}$ by means of the replacement:$x - \alpha = \frac{1}{z}$. Then the equation defining the relationship between algebraic functions and integrals of the form $\int \frac{(x-\alpha)^m dx}{\sqrt{R}}$ can be written as follows:

$$\int \frac{dx}{(x-\alpha)^m \sqrt{R}} = b_0 \int \frac{dx}{\sqrt{R}} + b_1 \int \frac{dx}{(x-\alpha)\sqrt{R}} + b_2 \int \frac{dx}{(x-\alpha)^2 \sqrt{R}} + \left(\frac{a_0}{(x-\alpha)^2} + \frac{a_1}{(x-\alpha)^3} + \ldots + \frac{a_{m-3}}{(x-\alpha)^{m-3}}\right)\sqrt{R}$$

From the above, we conclude that $\int \frac{Pdx}{\sqrt{R}}$, where P is some rational function, is expressed by algebraic functions and transcendental ones of the form:
$\int \frac{dx}{\sqrt{R}}, \int \frac{xdx}{\sqrt{R}}, \int \frac{x^2 dx}{\sqrt{R}}, \int \frac{dx}{(x-\alpha)\sqrt{R}}$.
Making changes $x = \frac{p+qy}{1+y}, R_1 = c(y^2 \pm a)(y^2 \pm b), \lambda = \frac{p-\alpha}{q-\alpha}$, where p, q, a, c are arbitrary constants in the corresponding algebraic expansions, we go over to integrals of the form:

$$\int \frac{dy}{\sqrt{R_1}}, \int \frac{y^2 dy}{\sqrt{R_1}}, \int \frac{dy}{(y^2-\lambda^2)\sqrt{R_1}}.$$

We write the last integral in terms of trigonometric functions:

$$\int \frac{f(\sin^2\varphi)d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}$$

where f is a rational function, k k is a real number $> 1$and φ is a real angular value.

Using the substitution $y^2 = \frac{A + B\sin^2\varphi}{C + D\sin^2\varphi}$ and sequentially considering the possible signs in the expression $R_1 = c(y^2 \pm a)(y^2 \pm b)$, we obtain:
$\int \frac{y^2 dy}{\sqrt{R_1}} = \int \frac{\alpha + \beta\sin^2\varphi}{\gamma + \delta\sin^2\varphi} \cdot \frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}$, where $\alpha, \beta, \gamma, \delta, k$ are constants, such that $k^2 > 1$.

Such integrals Abel called modular functions. To shorten, we introduce the following notation:
$\sqrt{(1-k^2\sin^2\varphi)} = \Delta(k,\varphi)$, or simply $\Delta\varphi$.The arc φ is the amplitude, the constant k is the modulus, the value$k' = \sqrt{1-k^2}$ is an additional

module.
Taking into account the introduced designations, the elliptic integral or modular function can be written as follows:

$$\int \frac{y^2 dy}{\sqrt{R_1}} = \int \frac{\alpha + \beta \sin^2 \varphi}{\gamma + \delta \sin^2 \varphi} \cdot \frac{d\varphi}{\Delta\varphi}$$

Abel considered this integral as a function of φ:

$$H(\varphi) = \int \frac{\alpha + \beta \sin^2 \varphi}{\gamma + \delta \sin^2 \varphi} \cdot \frac{d\varphi}{\Delta\varphi}.$$

Figure 3: Niels Henrik Abel (1802-1829)[2].

Now let's look at some of the properties of this function. First, this function is odd: $H(-\varphi) = -H(\varphi)$; secondly, it is periodic: $H(\varphi) =$

[2]Нильс Хенрик Абель. From Digital collection of paintings by Gørbitz, Johan (2022).

$H(n\pi \pm \psi) = 2nH(\frac{\pi}{2}) \pm H(\psi), 0 < \psi < \dfrac{\pi}{2}.$
Further, working with the function $H(\varphi)$, namely, looking over all possible values of the coefficients $\alpha, \beta, \gamma, \delta, k$ we come to the conclusion that all elliptic functions are reduced to three:

$$F(\varphi) = \int\limits_0^{\varphi} \frac{d\varphi}{\Delta\varphi} - \text{ an elliptic function of the first kind,}$$

$$E(\varphi) = \int\limits_0^{\varphi} \Delta\varphi \cdot d\varphi - \text{ an elliptic function of the second kind,}$$

$$\mathbf{P}(n, \varphi) = \int\limits_0^{\varphi} \frac{d\varphi}{(1 + n\sin^2\varphi)\Delta\varphi} - \text{ an elliptic function of the third kind.}$$

$n = \dfrac{\delta}{\gamma}$ is an elliptic function parameter $\mathbf{P}(n, \varphi)$. This parameter can be imaginary, and then a special class of functions is distinguished – hyper-elliptic or ultra-elliptic.

Sometimes functions $F(\varphi)$, $E(\varphi)$, $\mathbf{P}(n, \varphi)$ are replaced by three others, in which the variable is: $\sin(\varphi) = x$.

$$T_1(x) = \int\limits_0^{x} \frac{d\,x}{\Delta x},$$

$$T_2(x) = \int\limits_0^{x} \frac{x^2 d\,x}{\Delta x},$$

$$T_3(n, x) = \int\limits_0^{x} \frac{d\,x}{(1 + nx^2)\Delta x},$$

$$\Delta x = \sqrt{(1 - x^2(1 - k^2x^2))}.$$

CarL Jacobi (1804 - 1851) additionally introduced a notation for inverse functions. If α is the value of the function $F(\varphi)$, then to denote the inverse function we use the notation

$$\varphi = am(\alpha)$$

[3]Карл Якоби

Figure 4: Carl Jacobi (1804 - 1851)[3].

Then α is called the argument of its amplitude φ. The corresponding trigonometric dependencies of φ on its amplitude will be

$$\sin am(\alpha),\ \cos am(\alpha),\ \tan am(\alpha)\ \ etc$$
$$\Delta\varphi = \Delta am(\alpha)$$

The general property of all elliptic functions was discovered by Giulio Carlo de Toschi di Fagnano (1682-1766) and fully proved by Leonard Euler ("Institutionum calculi integralis. V.I."). It consists of the following:

If $\psi(x)$ is a transcendental function, and $\dfrac{d\psi(x)}{dx}$ is algebraic, then one can find such an algebraic relationship between the particular values $x = x_1, x_2, \ldots x_n$, in which the

$$m_1\psi(x_1) + m_2\psi(x_2) + \ldots m_n\psi(x_n),$$

where $m_1, m_2, \ldots m_n$ are commensurate positive or negative numbers, can be expressed or constant, or an algebraic function with respect to $x_1, x_2, \ldots x_n$, or a logarithmic function of these quantities.

Abel extended this property to all transcendental functions with algebraic differentials. This generalization is the central theorem of the theory of elliptic functions and is called Abel's theorem. The formulation and proof of this theorem was presented by Abel in *Précis d'une théorie des fonctions elliptiques.* This work of Abel was published posthumously in *Mémoires présentés par divers savants á l'Académie des sciences de l'Institut national de France et imprimés par son ordre* T.VII, 1841.

3. Abel's theorem. In the section we provide the representation of elliptic functions in terms of theta functions (v. Сомов, Осип Иванович (1850), Psheborskiy (1895), Hurwitz (1932)). Let an elliptic function $T_3(x) = \int \dfrac{d\,x}{(1-\dfrac{x^2}{a^2})\Delta x}$ be given, where $\Delta x = \sqrt{(1-x^2(1-k^2x^2))}$, $k^2 > 1$ and a is a constant. The following decomposition is allowed:

$$\psi(x) = A(x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2)\ldots(x^2 - x_\mu^2),$$

where the function $\psi(x)$ is the composition of two functions $\varphi(x)$ and $f(x)$:

$$\psi(x) = [f^2(x)] - [\varphi^2(x)](\Delta x)^2$$

$f(x)$ and $\varphi(x)$ are entire functions with undefined coefficients, $f(x)$ is even, $\varphi(x)$ is odd. Then the sum of the values of the function $T_3(x)$ for $x = x_1, x_2, \ldots x_\mu$ is expressed by the logarithmic function

$$T_3(x_1) + T_3(x_2) + \ldots T_3(x_\mu) = const - \frac{a}{2\Delta a} \cdot \log[\frac{f(a) + \varphi(a)\Delta a}{f(a) - \varphi(a)\Delta a}].$$

We briefly outline the proof of Abel's theorem.
Take one of the quantities for x: $x_1, x_2, \ldots x_\mu$, then

$$\psi(x) = [f^2(x)] - [\varphi^2(x)](\Delta x)^2 = 0$$

This equation determines the dependences of x on the coefficients of the functions $f(x)$ and $\varphi(x)$.

Considering these coefficients as variables, we write down the differential for this equation:

$$\psi^{'}(x)dx + \delta\psi(x) = 0,$$

, where $\psi^{'}(x)$ is the derivative of $\psi(x)$ with respect to x, and $\delta\psi(x)$ is the differential of the same function with respect to the variable coefficients of the functions $f(x)$ and $\varphi(x)$. In this differentiation, x will be considered, respectively, a constant value.

$$\delta\psi(x) = 2[f(x)\delta f(x) - \varphi(x)\delta\varphi(x)](\Delta x)^2$$

The corresponding functions $f(x)$ and $\varphi(x)$ can be expressed from the equation:

$$\psi(x) = [f^2(x)] - [\varphi^2(x)](\Delta x)^2 = 0. \text{ We get:}$$
$$f(x) = \pm\varphi(x)\Delta x, \varphi(x)(\Delta x)^2 = \pm f(x)\Delta x.$$

Hence

$$\delta\psi(x) = -2[\varphi(x)\delta f(x) - f(x)\delta\varphi(x)]\Delta x = -\Theta(x)\Delta x$$

Introducing the notation:

$$\Theta(x) = 2[\varphi(x)\delta f(x) - f(x)\delta\varphi(x)]$$

,we get:

$$\psi^{'}(x)dx = \Theta(x)\Delta x \Rightarrow \tfrac{dx}{\Delta x} = \frac{\Theta(x)}{\psi^{'}(x)} \Rightarrow T_3(x) = \textstyle\int \frac{\Theta(x)}{(1 - \frac{x^2}{a^2})\psi^{'}(x)}$$

Substituting in this expression $x_1, x_2, \ldots x_\mu$ instead of x, and summing the results (denoting $\sum$), for abbreviation, we get:

$$\sum T_3(x) = \int \sum [\frac{\Theta(x)}{(1 - \frac{x^2}{a^2})\psi'(x)}]$$

The integrand is a rational symmetric function of the roots of the equation
$\psi(x) = [f^2(x)] - [\varphi^2(x)](\Delta x)^2 = 0$, therefore it can be expressed by a rational function of its coefficients and, therefore, by a rational function of the coefficients of the functions $f(x)$ and $\varphi(x)$.

Thus:

$$\sum T_3(x) = \int \frac{\Theta(x)}{(1 - \frac{x^2}{a^2})\psi'(a)} = \int \frac{a\Theta(a)}{2\psi(a)}.$$

Substituting here the values of the functions $\Theta(x)$ and $\psi(x)$, we have

$$\sum T_3(x) = \frac{a}{2}\int \frac{\varphi(a)\delta f(a) - f(a)\delta\varphi(a)}{[f^2(a)-\varphi^2(a)](\Delta a)^2}$$

Now let's integrate using the replacement $\dfrac{\varphi(a)\Delta a}{f(a)} = z$. We get

$$\sum T_3(x) = -\frac{a}{2\Delta a}\int \frac{\delta z}{1-z^2} = Const. - \frac{a}{4\Delta a}\log(\frac{1+z}{1-z})^2.$$

So finally,

$$\sum T_3(x) = Const. - \frac{a}{4\Delta a}\log[\frac{f(a)+\varphi(a)\Delta a}{f(a)-\varphi(a)\Delta a}]^2$$

The proved theorem gives similar formulas for elliptic functions of the first and second kind:

$$T_1(x) = \int \frac{d\,x}{\Delta x}, T_2(x) = \int \frac{x^2 d\,x}{\Delta x.}$$

The function $T_1(x)$ is the value of the function $T_3(x)$ at $a=\infty$ and $T_2(x)$ is the value of the function $a^2[T_3(x)-T_1(x)]$ $a=\infty$.

$$a=\infty \Rightarrow \frac{a}{4\Delta a} = 0, \log(\frac{1+z}{1-z})^2 = \log(1) = 0, \Rightarrow \sum T_1(x) = Const,$$
$$\sum T_2(x) = Const. - \frac{a^3}{4\Delta a}\log[\frac{f(a)+\varphi(a)\Delta a}{f(a)-\varphi(a)\Delta a}]^2\Big|_{a=\infty}.$$

Some of the values $x_1, x_2, \ldots x_\mu$ are taken arbitrarily, the rest are determined from them.

Then, working with logarithms, Carl Jacobi gets two theta functions and decomposes them into rapidly converging series.

Proving Abel's theorem, Carl Jacobi introduced the famous Θ functions, with the help of which hyperelliptic integrals can be represented by algebraic expansions or rapidly converging series. The latter representation, using series, is very important in the problem of the rotation of a rigid body about a fixed point. Using these expansions, O.I. Somov obtained the kinematic parameters of motion in the problem of the rotation of a rigid body about a fixed point in the case of an initial impact. The presented solution is very important in practical problems of orientation and control of objects of mechanical motion. Next, we analyze the course of Somov's reasoning regarding the problem of the rotation of a rigid body (v. Leimanis (1965)).

4. The problem of the rotation of a rigid body. Solution of the problem of the rotation of a rigid body about a fixed point in the case of an initial impact was presented by Сомов, Осип Иванович (1850).

For the first time, the problem of the rotation of a rigid body, and at once in a general form, was posed by Euler in 1758. In his work "Theory of motion of rigid bodies" he considered the case of motion of a rigid body around a fixed point (pole). In this case, the body has three degrees of freedom. Leonhard Euler calls these three parameters angles: precession - ψ, proper rotation - φ, nutation - θ, which uniquely determine the position of the moving frame of reference rigidly connected to the body relative to the fixed frame of reference. When a rigid body rotates, Euler's angles change, being some functions of time, which he derived in his work "The Discovery of a New Principle of Mechanics" (1750). Kinematic equations (relationships between the angular velocities of the body and the parameters of motion) were obtained. Further, Euler establishes the relationship between the parameters of motion and the forces acting on the body - dynamic equations.

The next step forward was soon taken by Joseph Louis Lagrange (1736-1813). An essentially new achievement of Lagrange in this problem was the formulation of the problem of the motion of a rigid body, which was later called the "Lagrange case": the point of support or suspension does not coincide with the center of gravity of the body. Lagrange brought the problem of the rotation of a rigid body about a fixed point to quadratures. In 1773 he investigated the rotation of a rigid body of some shape about a fixed point, having considered and refined Euler's solution to the problem of the rotation of a body without the action of external forces.

An essential step in the development of rigid body dynamics was made by Carl Jacobi (1854–1851) by introducing elliptic functions. With the help of these functions, time becomes an independent variable and the main parameters of the rotation of a rigid body are single-valued functions of time. C. Jacobi's work on elliptic functions was carried out by him in 1849, and was published posthumously in his second volume of works by the Berlin Academy of Sciences in 1882. Thus, the mathematical apparatus of elliptic functions created by Jacobi allowed O.I. Somov in 1850 brilliantly solved the problem of the rotation of a rigid body in the event of an initial impact.

In 1850, O.I. Somov (1815–1876) solved the problem of the rotation of a rigid body about a fixed point in a new setting, different from Euler and Lagrange: for the case when the motion occurs only from the initial impact.

The solution is based on differential equations of motion:

$$\begin{cases} dt = \dfrac{A}{B-C} \cdot \dfrac{dp}{qr}, \\ dt = -\dfrac{B}{A-C} \cdot \dfrac{dq}{pr}, \\ dt = \dfrac{C}{A-B} \cdot \dfrac{dr}{pq}. \end{cases}$$

In this case, the body has three degrees of freedom and begins to rotate from the initial impact. Further, there are no shock loads, only gravity acts. A, B, C are the moments of inertia on the axes parallel to the main axes of the rotating body. The position of these axes is determined by the Euler angles: ψ, φ, θ (precession, proper rotation, and nutation). Accordingly, p, q, r are the angular speeds of rotation relative to these axes. Performing elementary algebraic operations with the equations written above, we obtain a system of fundamental integrals:

$$A^2p^2 + B^2q^2 + C^2r^2 = l^2 (l^2 - \text{ moment of initial impact}),$$
$$Ap^2 + Bq^2 + Cr^2 = h (h - \text{kinetic energy}).$$

To determine the angle ψ we have: $d\psi = -\dfrac{Ap^2 + Bq^2}{A^2p^2 + B^2q^2} \cdot l \cdot dt$. Then changing the angle is $\psi = -l \int \dfrac{Ap^2 + Bq^2}{A^2p^2 + B^2q^2} \cdot dt$. Further, integrating, we get:

$$\psi = -\frac{l}{A} \cdot \int_{t_0}^{t} dt - \frac{l(A-B)}{A} \cdot \int_{t_0}^{t} \frac{Ap^2 + Bq^2}{A^2p^2 + B^2q^2} \cdot dt.$$

To derive the angular velocities, Somov uses the apparatus mentioned above in the theory of elliptic functions. Thus, the instantaneous velocities during the rotation of a rigid body after the initial impact can be written as follows.

$$\begin{cases} p = -\dfrac{l}{A} \sin\theta \sin\varphi = -\sqrt{\dfrac{l^2 - Ch}{A(A-C)}} \cos am(u), \\ q = -\dfrac{l}{B} \sin\theta \cos\varphi = -\sqrt{\dfrac{l^2 - Ch}{B(B-C)}} \sin am(u), \\ r = \dfrac{l}{C} \cos\theta = \pm\sqrt{\dfrac{Ah - l^2}{C(A-C)}} \cdot \Delta am(u). \end{cases}$$

where $u = n(t - t_0), n = \sqrt{\frac{(B-C)(Ah-l^2)}{ABC}}$.

Substituting the expressions for the velocities into the angle ψ, Somov receives the following:

$$\psi = -\frac{l}{An} \cdot u - \frac{l(A-B)(A-C)}{A^2(B-C)n} \int_0^u \frac{\sin^2 am(u)du}{1 + \frac{C(A-B)}{A(B-C)} \sin^2 am(u)}$$

The last integral is an elliptic function of the third kind with an imaginary parameter. Writing it down in terms of the theta function $\theta(x)$, we obtain an expression for the precession angle ψ:

$$\psi = -nu \pm \frac{i}{2} \cdot ln \frac{\theta(u-ai)}{u+ai}.$$

5. Conclusion. The theory of elliptic functions was brilliantly continued by Karl Weierstrass (v. Psheborskiy (1895)). The elemen-

Figure 5: Karl Weierstrass (1815-1897)[4].

tary exposition of the elliptic function was presented by Cayley 1876.

[4] Карл Вейерштрасс

Working under the guidance of the famous professor Christoph Gudermann (1798–1852), the young scientist became interested in the research of Abel and Jacobi in the field of elliptic functions. Weierstrass managed not only to penetrate but also to solve the problems that Abel and Jacobi had only indicated. So, he found a representation of a modular function in the form of a quotient of two series, and generalized this representation to other well-known elliptic functions. He managed to do this in the summer of 1840, and in the fall of the same year he defended his work: *Ueber die Entwickelung der Modular-Functionen*. Part of this work was included in his memoirs on Abelian functions, published in the 52nd issue of Crelle's Journal[5] (v. Gudermann (1843, 1842, 1840, 1840, 1840, 1839) or volume I of his Mathematische Werke).

Weierstrass replaced all particular forms of the θ-function with one $\wp$-function Klein (1897). In 1871, Weierstrass simplified the system of Jacobi elliptic functions by introducing, instead of three theta functions, the one, which has complex time as its argument.

Weierstrass's research in the theory of elliptic functions drew the attention of German scientists, and in 1856 he was invited to the Berlin University as an extraordinary professor at the Department of Pure Mathematics. In 1857 he was elected a member of the Berlin Academy of Sciences. Later, Weierstrass, generalizing and refining the conclusions of Abel and Jacobi, showed that the conditions for the integrability of an elliptic integral in logarithmic functions can be derived if such integrals are divided into three classes of the first, second and third kind.

Weierstrass reported the main results related to hyperelliptic integrals either in his lectures or in letters to colleagues. Studying S.V. Kovalevskaya's works, we see that she tried to use Weierstrass methods, placed in the letters to her, in relation to the problem of the rigid body rotation about a fixed point. Carefully studying her work, it becomes clear that she did not succeed. The St. Petersburg mathematician and mechanic Professor Somov brilliantly dealt with the application of hyperelliptic functions in this problem. By 1851, Somov gave the first generalized solution to the problem of rotating a body around a fixed point. Somov obtained a solution to the problem of the rotation of a rigid body about a point after an initial impact by integrating the differential equations of motion using Jacobi elliptic functions of the third kind with an imaginary parameter. Somov's solution showed that the

[5] Weierstraß (1856). Crelle's Journal, or just Crelle, is the common name for a mathematics journal, the Journal für die reine und angewandte Mathematik

main parameters of motion are expressed through the composition of elliptic functions of the simplest form and, introducing them, the problem of the rotation of a rigid body relative to a fixed point is reduced to the simplest elements. Therefore, further investigation of the problem of the motion of the top was continued on the complex plane; the direction cosines during the rotation of the body were obtained in the form of partialθ or σ -functions. Quaternion expressions, discovered by W. Hamilton (1805-1865) in 1843 and developed in vector analysis by J.W. Gibbs (1839-1903) in the 1880s, obtained clarity in Somov's works for the kinematics of motion in the problem of rotation of a rigid body about a fixed point.

Thus, we see how the mechanical problem posed by Euler and developed in the work of Lagrange received its mathematical embodiment in the theory of elliptic functions. The development of both theories has mutually enriched their content.

The chronology research by the author of the article is limited to 1888. Somov's 1850 work on the spinning top provides a general solution to the classical Euler problem. In 1855, on the basis of this work, he proved the gyroscopic effect (A=2C). Thus, by 1888 Somov's task was fully completed. An analytical substantiation of the gyroscopic effect can be presented in the next article. Thus, two works on the legacy of Somov will give a complete picture.

Gratitude: In conclusion, I express my sincere deep gratitude for the formulation of the problem, great attention, and constant assistance in the development of problems to the doctor of physical and mathematical sciences, professor of the department "Mathematics" of the St. Petersburg State University of Architecture and Civil Engineering Galina I. Sinkevich.

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Historia zastosowań teorii funkcji eliptycznych do rozwiązania problemu obrotu ciała sztywnego w XIX wieku

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Streszczenie. Praca poświęcona jest zastosowaniom koncepcji funkcji eliptycznej do problemów mechaniki, głównie do rozwiązania problemu obrotu ciał sztywnych. Naszą uwagę skupimy na najbardziej znaczących wynikach w tym zakresie zawartych w pracach Abela, Jacobiego, Weierstrassa i Somova. Wychodząc z dowodu twierdzenia Abela pokazano reprezentację funkcji eliptycznych w kategoriach funkcji theta (θ). Artykuł zawiera szkic rozwiązanie klasycznego problemu obrotu ciała sztywnego wokół punktu stałego za pomocą funkcji hipereliptycznych.

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Słowa kluczowe: Funkcja eliptyczna, Jacobi, Abel, Somov.

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