Rolle's Theorem and Bolzano-Cauchy Theorem : A View from the End of the 17th Century until K. Weierstrass' epoch

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ABSTRACT

We discuss the history of the famous Rolle's theorem "If a function is continuous at $[a, b]$, differentiable in (a, b), and $f(a) = f(b)$, then there exists a point c in (a, b) such that $f'(c) = 0$", and that of the related theorem on the root interval, "If a function is continuous on $[a, b]$ and has different signs at the ends of the interval, then there exists a point c in (a, b) such that $f(c) = 0$".

Key words : Rolle's Theorem, Bolzano–Cauchy Theorem.

INTRODUCTION

Contemplating over the 19th century reforms in analysis bring to mind mainly two names, A. Cauchy and K. Weierstrass. Actually, quite a few mathematicians form part of any meaningful history of the topic during the era. Some of them lived before, while others were their contemporaries, and without due reference to them the history of the topic would not only lack certain important colours, but also the thoroughness needed in a historical analysis.

Mathematician and theologist Bernard Bolzano was one of the ingenious messengers of the ideas that brought the reform. Many fundamental ideas which were later developed by Cauchy, Weierstrass, Cantor and Dedekind, were originated by him. These include the notion of the least upper bound (1817), realization of the need for developing a theory of real numbers (1830s) and set-theoretic understanding of numbers.[1]

[1] Bolzano's ideas gained popularity in Germany thanks to Hermann Hankel who in 1870 published the above mentioned work of Bolzano in Tubingen and popularized his other works. Otto Stolz's work devoted to Bolzano [Stolz, 1881] is of interest as well. Cauchy knew Bolzano personally and used his ideas in his own works [Grattan-Guinneess, 1970, H. Freudenthal, 1971, Sinkevich G. 2012a].

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In 1817, Bernard Bolzano wrote a work entitled *Purely analytic proof of the theorem that between any two values giving results of opposite sign, there lies at least one real root of the equation.* [Bolzano, 1817, 1996]. Bolzano attributed the importance of the key property of a continuous function to this theorem and considered its genesis. Let us follow his lead.

Before the 17th century, they used geometric images of curve crossings to find roots of algebraic equations, while the interval in which the roots lay was determined based on ratio analysis. For example, Newton wrote in his *Universal Arithmetics:* "Should you want to find a limit which cannot be exceeded by any root, find a sum of squared roots and take the square root of this sum. This square root will be larger than the largest equation root" [Newton, 1707, p. 265]. In the middle of the 17th century, they used the method of root localization with the help of an auxiliary equation. As a rule, they would look for positive roots. In order to make an auxiliary equation, exponents of variables were reduced by a unity, and each factor was multiplied by the former (Johann Hudde, 1658)[2]. Later, this operation was defined as differentiation of a polynomial (Isaac Newton and Gottfried Leibnitz).

1690, Rolle and his Method of Cascades

Michel Rolle (1652 - 1719) was born in France in a small town, Ambert, in the province Auvergne, in the family of a shoemaker. He came to Paris when he was 23 to work there as an enumerator. He was so successful in self-education that in 1682 managed to solve a difficult problem posed by Jacques Ozanam (1640 - 1717): "Find four numbers such that the difference of any two is a square, and the sum of any two of the first three is also a square"[3]. Ozanam himself believed that each of these numbers would consists of at least 50 digits. However, Rolle found such numbers, each containing no more than 7 digits. This brought him considerable mathematical repute. He was invited to tutor the son of the Minister of War, got an appointment in the War Ministry, a pension from Louis IV, and in 1685, he became

[2] A reconstruction of Hudde's method was provided by A.P. Yushkevich in his comments to the translated work of L'Hôpital entitled "Infinitesimal calculus" [L'Hôpital, G.F. 1935, p. 400].

[3] Trouver quatre nombres tels que la différence de deux quelconques soit un carré, et que la somme de deux quelconques des trois premiers soit encore un carré. In MacTutor by J.J. O'Connor and E.F. Robertson the translated text has missed the part "of any two".

member of the Royal Academy of Science (as a student of an astronomer), and from 1699, its welfare recipient (as a geometrician, that is to say, mathematician).

Rolle dealt with algebraic issues: Diophantine analysis, solving algebraic equations. He was instrumental in popularising R. Recorde's algebraic symbolism and introduced the notation $\sqrt[n]{x}$.

Rolle is known for his aggressive criticism of differential calculus and method of Descartes, for what he considered as lack of adequate rationale. French mathematicians P. Varignon and J. Saurin refuted most of Rolle's arguments; in 1705, the Academy concluded that he was wrong, to which he later agreed. However, the episode had the salutary effect of causing Leibnitz to treat differential calculus more rigourously.

In 1690, Rolle published *A Treatise on Algebra* [Rolle, 1690] devoted to solving of Diophantine and algebraic equations with arbitrary exponents. It contained many innovative ideas related to the method of Descartes. Method of Cascades as one of the methods [Rolle, 1690, p. 124 -152] was based on the idea that roots of the initial equation are separated by roots of an auxiliary derived equation. Roots of the auxiliary equation can also be separated with the help of another auxiliary equation, etc. Forming cascades, we descend to a linear equation; and once we solve it, we ascend back to the original question and get a solution for it. Rolle published the justification of his method a year later in a small work entitled *Démonstration d'une méthode pour ré soudre les égalités de tous les degrés* (Justification of a method solving equations of all degrees), Paris, 1691.

A good exposition of the method is found in [Shain 1937, Janovskaja, 1947, Washington].

In addition to this method for solving algebraic equations, Rolle offers four more methods, as well methods for solving indefinite equations, and a method of finding the common divisor of polynomials.

Using coefficients, Rolle selected limits between which the roots lay. In the method of cascades, although without differential calculus terminology, he used the principle of partitioning of roots of the polynomial by roots of its derivative, and the existence of roots was checked by the difference in the signs of the polynomial at the ends of the interval. In 1691, in his work devoted to the justification of the method of cascades, Rolle demonstrated that the values of the derivative (i.e. the derived polynomial) for two adjacent (single) roots of the integral polynomial have different signs [Rolle, 1691, p. 47].

The notion of a function was only evolving in the 17th century; there was no notion of graph of a function, or geometrical locus at that time. Therefore, the concept of the root as a point where the function graph crosses the axis did not exist as yet.

This scenario is noticeable in Michel Rolle's works. He concluded existence of a root in the interval by determining signs of the polynomial involved in the equation, at the ends of an interval. If the signs were different, a root lay within the interval. Rolle narrowed the interval checking the sign of the polynomial in various interior points of the interval. Thus, Michel Rolle became the father of two theorems in Analysis: the theorem on the root interval currently known as the Bolzano - Cauchy theorem and the usual Rolle's theorem, stating that roots of a continuous function are separated by roots of its derivative.

A detailed presentation of the history of the method can be found in [Barrow-Green 2009]. The historical analysis of Rolle's theorem in [Barrow-Green], involves also other names, Odoardo Gherli (1771), François Moino (1840), Orly Terquem (1844), Homersham Cox (1851), Joseph Liouville (1864), Joseph Serre (1868), and Pierre-Ossian Bonnet (1868), but the historical line through Bolzano - Cauchy - Weierstrass - Cantor is missing.

The primary source of Rolle's biography is *Éloge de M. Rolle,* written by B. Fontenelle [Fontenelle, 1719], a contemporary of Rolle, and Secretary of the Paris Academy of Sciences. Rolle's biography and Rolle's method and its history are described in greater detail [Sinkiewich, 2013] (in Russian).

1707, Rolle and Newton

Before Rolle, approximate solution of algebraic equations was achieved by graphic methods, i.e. by curve crossing and with the help of simple iterations. Rolle was probably the first to form the notion of the root interval by comparing signs of the respective polynomial.

The geometric formulation of the problem did not correspond to the search of the curve crossing with the axis. Instead, it was the search of the points of intersection of two curves. Therefore, an image of a graph with ordinates of different sign at the ends of the interval could not appear in terms of algebra. Isaac Newton (1642 - 1727), for example, imagined variables to be time variant, not changing against each other [Mordukhaj-Boltovskoy, 1952]. The concept of a curve as a geometrical locus of an equation[4] first appeared in the work of L'Hôpital devoted to conic sections; thereafter, it was developed by Euler, and the general approach emerged only in the 19th century.

Newton described his tangent method in his works entitled *Analysis of equations with an infinite number of members*[5] and *Method of fliuxions and infinite series*[6]. This method was also stated in the book of 1685 by J. Wallis *A Treatise of Algebra both Historical and Practical*. In 1690, J. Raphson's (1647/48-1715) treatise was published in England. It was entitled *Analysis aequationum universalis* [Raphson, 1702] and contained an improved statement of Newton-Raphson method or tangents method[7]. In 1707, Newton's *Arithmetica Universalis* (Universal Arithmetic) was published. It contained equations numerical solution methods.

While using the tangents method, Newton did not consider signs of the function at the ends of the interval, until Rolle's treatise appeared, which may be seen from Newton's works published before 1690, e.g. in his *Method of Fluxions* of 1671 [Newton, 1736.]. To choose a starting point for the root calculation procedure, Newton used the method of false position and the so-called 'Newton parallelogram' or 'Newton polygon'.

The Paris Academy and the Royal Society of London exchanged academic literature. No doubt, Newton would have received Rolle's treatise. Moreover, he

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4 This concept should be distinguished from that of a geometrical locus of points which possess a given property (e.g. circumference) which appeared in the ancient world.

5 *De analysi per aequationes numero terminorum*, manuscript of lectures Newton read at the university, was written in Latin in 1669 and published in 1711.

6 *De methodis et serierum*, 1671, translated into English and published as *Method of Fluxions* in 1736.

7 While Newton considered the sequence of approximating polynomials, Raphson already considered successive iterations of the variable.

included the statement of his method in his 1707 publication of *Universal Arithmetics* [Newton, 1707, p. 267-270] without mentioning Rolle's authorship though. But it was only after 1707, when Rolle's work appeared, that Newton first started checking signs of polynomials at the ends of the interval in his *Universal Arithmetics*. The method of narrowing the interval containing the root by checking the sign of the polynomial at a certain interior point (not necessarily the mid-point) first occurred in Rolle's works. Bolzano formalized it 117 years later as half-interval method. It may be noted that Newton treated all functions under consideration to be continuous by definition, while Rolle dealt with only polynomials, which are of course continuous functions.

The Newton method and use of Maclaurin series expansion were more popular among continental mathematicians. In 1740, T. Simpson provided a summary of the Newton method in his work entitled *Essay on several subjects in speculative and mixed mathematics*[8] [Simpson, 1740].

1696, L'Hôpital

In 1696, the first textbook on analysis was published in Paris. It was *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* [L'Hospital, 1696] by marquis G.F. de L'Hôpital (L'Hospital l, 1661-1704) stating lectures of I. Bernoulli (1667 - 1748). It was for the first time that differential and integral calculus was stated there and the notions of abscisse, ordinate, coordinates, and geometrical locus were used. However, L'Hôpital achieved the simplicity and intelligibility of presentation by omitting argumentation: "I am sure that in mathematics only conclusions are of use and that books describing but details or particular suggestions only make those who write them and those who read them waste time" [L'Hôpital, 1935, p. 57]. But he provided the geometrical meaning of a derivative, relation between the function increasing or decreasing and the sign of the first derivative, the necessary condition of the extremum. One would also find the following reasoning there: "No continuously increasing or decreasing positive value[9] can turn into negative without passing through infinity or zero,

[8] Simpson already used derivatives for iterations.

[9] L'Hôpital means subtangent.

more specifically: through zero when it is first decreasing and through infinity when it is first increasing. This implies that the differential of the largest or the least value must equal zero or infinity. It is easy to understand that a continuously decreasing positive value cannot turn into negative without passing through zero; however, it is not so obvious that in the case of an increase it has to pass through the infinity" [L'Hôpital, 1935, p. 130 -132].

This book opened the initial period of development of analysis where all functions were continuous, as they were algebraic functions, and all analytical statements were based on geometrical ideas. The rules of differential calculus of the 17th-18th century were defined for algebraic functions only. Formulas of derived transcendental functions would appear later in Euler's and Cauchy's works.

1708, Rolle's Method in Treatise by Reynaud

C.R. Reynaud (1656-1728), French preacher and professor of philosophy, was familiar with works of Hudde, Descartes, Rolle, Newton, Leibnitz, Bernoulli, and L'Hôpital. Being aware of the reproaches in relation to the insufficiency of argumentation and lack of methodical statement of modern mathematics, he made it his crusade to deliver a complete course of analysis, algebra and geometry with cross-references and proofs. In 1708, his book entitled *Demonstrable analysis* [Reynaud, 1708, 1736] in two volumes was published in Paris describing results of the above mentioned mathematicians. I would note that before that time, only geometrical statements were proved mathematically, while in algebra and evolving analysis they were only illustrated with examples.

The first volume of *Demonstrable Analysis* was devoted to algebraic issues and the second one to differential and integral calculus, where the author tried to prove most of the statements, provided quite a lot of examples, not only mathematical ones but from mechanics and astronomy as well. Notes to the second edition of 1736 were written by Varignon.

A professional preacher, Reynaud had good command on presentation and was good at selecting terms not only in Latin but also in French. His manner of presentation compared favourably to the complicated language of Michel Rolle, not only in terms of speech melody in general, but in terms of consistency of argumentation and appropriateness of definitions as well. One can feel an experienced teacher in his techniques. First, Reynaud considered linear equations forming equations of the highest degrees by multiplying binomials, proceeding up to equations of the sixth degree. He described solutions to equations involving not only numerical powers but also algebraic (letter) symbols, both in radicals and approximately; he provided the fundamental theorem of algebra—the theorem on the value of the residue of division of a polynomial by a binomial[10] [Reynaud, 1736, T. I, p. 270-271]. His demonstrations included verbal proof accompanied by demonstration of particular cases and examples. In algebraic equations, Reynaud distinguished between cases with single, multiple, positive, negative, whole, fractional, non-measurable, and imaginary roots.

Reynaud devoted a large section [Reynaud, 1736, T. I, p. 269-375] to the method of Rolle: "The sixth book explains and proves the method of finding the values which constitute limits of the unknown in the exponential equation (Mr. Rolle was the author of this method) and provides some solution methods using these limits; the roots here can be found with any degree of accuracy that one may be wish" [Reynaud, 1736, T. I, p. XII].

While developing Rolle's ideas, Reynaud introduced his own terminology. He called each auxiliary equation the roots of which are limits of roots of the previous one 'l'équation des limites' (equation of limits) and the boundaries of the interval which contains a root, 'limits of roots'. Reynaud defined the root interval which contains a real root based on the difference of signs in the left part of the equation on the limits of the roots; he described Rolle's step-up and step-down procedures.

The second volume of *Demonstrable Analysis* is devoted to differential and integral calculus. It contains a statement that a tangent to a curve (for conic sections) at a certain point is parallel to the diameter [Reynaud, 1736., T. 2, p. 176). Another statement contained in it reads as follows: if a sequence of values of a variable (e.g. subtangent[11]) is first positive and thereafter becomes negative, this means that it passes a certain point where its value equals zero or infinity [Reynaud, 1736., T. 2, p. 177].

[10] This theorem bears the name of E. Bezout (1730-1783). One can come across this very theorem in J. Raphson's (1648-1715) works.

[11] A subtangent is a projection of an interval of a tangent between the points of intersection with axis *OX* and tangency point on axis *OX.*

1727-1729, Rolle's Theorem in the works of Campbell and MacLaurin

Reynaud's *Demonstrable Analysis* was quite well known in England. G. Campbell[12] quoted it in 1727 in his work entitled *A Method for Determining the Number of Impossible Roots*[13] *in Adfected Aequations*. [Campbell, 1727/28]. Campbell translated French mathematical works into English and solved algebraic equations himself. In the above work, he paraphrased Rolle's procedure, engaged Fermat's rule for determining maxima and minima, and considered the case with the final quadratic equation with a negative discriminant. Then, judging from the change of signs of coefficients of the initial equation, it is possible to tell the number of imaginary roots or rather their least number. In his letter [MacLaurin, 1729] to the same journal, C. MacLaurin (1698-1746) argued against rigidity of the demonstration. MacLaurin formulated the theorem as follows: Roots of equation $x^n - Ax^{n-1} + Bx^{n-2}$ & c. =0 are limits of roots of equation $nx^{n-1} - (n-1)\,Ax^{n-2} + (n-2)\,Bx^{n-3}$ & c. =0" [MacLaurin, 1729, p. 88]. The change of signs of the coffiecients ensures n positive roots. This statement had already had the status of a theorem. MacLaurin also considered a more general type of an auxiliary equation used by Hudde in 1658.

1746, A.C. Clairaut

In 1746, a book by A.C. Clairaut (1713-1765), *Élémens d'Algèbre* [Clairaut, 1746], was published to tell about methods to solve algebraic equations. Much attention was paid to the method of Newton and MacLaurin there, but Rolle and his method were not mentioned.

1755, Rolle's Theorem in Euler

In 1755, St. Petersburg Academy of Sciences published L. Euler's (1707-1783) work entitled *Institutiones calculi differentialis* (Foundations of Differential Calculus). The trend towards convergence of algebra and analysis that we saw in Reynaud's treatise and discussions of the equation of the string by D'Alembert and Euler led to an

[12] This Scottish mathematician is only known for having argued against C. MacLaurin; he died in 1766.

[13] Imaginary ones.

extension of the notion of a function. Euler was proud that he did not have to turn to applied interpretation when stating analysis. He wrote in Chapter IX: "The notion of the equation can be traced to the notion of function" [Euler, 1755, p. 367]. Euler considered a polynomial there as a trivially continuous function which satisfied his concept of a continuous function as a function assigned by an integrated analytic expression. Euler repeated the above theorem of MacLaurin on roots of equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + dx^{n-4} -$ etc... = 0, separated by the roots of the auxiliary equation, i.e. by extreme values. It may be noted that both Maclaurin and Euler considered an equation which trivially had n real roots. The problem of determination of the number of imaginary roots set up by Campbell was considered by Euler in Chapter XIII. Euler summarized his reasoning as follows:

"However, it appears from the above that, although not all roots of the proposed equation may be real, nevertheless, there is always a maximum or a minimum between any two roots. As to the converse proposition, it is wrong in general, i.e. there may be no real root between any two maxima or minima. However, this conclusion may be made subject to an added condition that either the value of x will be positive and the other one, negative [. . .]. There is one value between two real roots of the equation at which the function has a maximum or minimum." [Euler, 1755, p. 435-436]. Euler's reasoning was based on the concept of continuous movement; he extended all properties of algebraic expressions to functions.

1797, Theorem on the Root Interval in S.F. Lacroix's *Élémens d'algèbre*

In 1797, the first edition of S.F. Lacroix's (1765-1843) *Élémens d'algèbre* was published in Paris. Lacroix was the author of courses of higher mathematics which were repeatedly reissued and well known in Russia of the 19th century.

In the *Élémens d'algèbre,* he provided the following theorem on the root interval:

"If there are two values which, being inserted into the equation instead of an unknown, will produce two results opposite in sign, we can conclude that the roots of this equation are between these two values and they are real". [Lacroix, 1830, p. 298].

In 1811, an authorized German translation of Lacroix's *Élémens d'algèbre* by M. Metternich (1747-1825), a professor of mathematics and physics of Mainz University [Lacroix, 1811], was published in Mainz. This book also contains the statement of the theorem on the root interval. This book was repeatedly reissued, in German, and widely used by German mathematicians.

1768, Kästner about selection of the root interval

Abraham Gottheld Kästner (1719-1800), a professor of mathematics and physics at Göttingen, was esteemed as a prominent specialist in teaching methods, on various issues relating to analysis. It may be noted that he considered irrational numbers as limits of sequences of rational numbers, before Cauchy. Kästner was in correspondence with Euler.

In 1768-69, Kästner wrote a course on *Fundamentals of Mathematics* ("Der mathematischen Angfangsgründe") in four volumes [Kästner, 1794]. It was an accomplished course in terms of methodology; it provided a good historical survey; and was repeatedly reissued. One can feel Euler's influence in the course. The course of Kästner was published in Russian in 1792-1803. There was no Rolle's method in the course of Kästner, but there is the theorem on root interval of a polynomial with a demonstration using geometric analogy. In the German publication of 1794, it reads as follows: "Theorem. If y is positive for $x = a$ and negative for $x = c$, then at least one value of $x = b$ will be found between a and c for which $y = 0$". [Kästner, 1794, p. 198]. Kästner had great influence on German mathematical education, and in particular Karl Weierstrass referred to his works.

1798, Lagrange on the Method of Rolle

In 1798, J.L. Lagrange (1736-1813) suggested his root isolating method based on the method of Rolle [Lagrange, 1798]. Lagrange asserted that roots of the initial equation were divided by roots of the derived equation and characterized by insertion of roots of the derived equation in the initial equation followed by determination of its sign: "Thus, these rules enable us to determine not only the number of real roots of the equation, but the boundaries within which they lie as well; and if you wish to constrain roots between a value that is larger than α_1 and less than v_1, an additional search needs to be carried out in accordance with the method stated in Chapter IV (No. 12) regarding the boundaries of positive roots of this equation. Please note that already Rolle knew the rules enabling one to find these limits and stated by us according to Newton and MacLaurin, which appears from Chapters V and VI of this Algebra" [Lagrange, 1798., p. 199].

1817, Bolzano and the theorem on root interval

Bernard Bolzano (1781-1848), Czech mathematician and philosopher, contributed quite a lot in the development of the notion of continuous and infinite. In his manuscript of 1817 entitled *Purely analytic proof of the theorem that between any two values which give results of opposite sign, there lies at least one real root of the equation* [Bolzano, 1817], he criticised the proofs of Kästner, Clairaut, Lacroix, Metternich, Rösling, Klügel, and Lagrange for the involvement of geometrical and physical images (time and movement, transfer) and the lack of analyticity in their reasoning, i.e. lack of understanding of the continuity as a mathematical notion[14]. Bolzano wrote: "The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the *x*-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical proposition. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry. No one will deny that the concepts of time and motion are just as foreign to general mathematics as the concept of space. We strictly require only this: that examples never be put forward instead of proofs and that the essence of a deduction never be based on the merely metaphorical use of phrases or on

[14] To tell the truth, as we could satisfy ourselves, neither Clairaut in the above work of 1746 nor Dr. Ch.L. Rösling (1774 - 1836) from Erlangen University in his book of 1805 entitled *Fundamentals of the theory of forms, differentials, derivatives and integrals of functions* [Rösling, 1805] turned to the method of Rolle and his theorem on the root interval. References to them provided by Bolzano are at variance with the topic concerned. [Bolzano, 1817, p.171].

their related ideas, so that the deduction itself would become void as soon as these were changed". [Bolzano, 1955, 1996, p. 172-174; 43].

Bolzano stated the law of continuity[15] as follows: "According to a correct definition, the expression that a function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits means just that: if x is some such value, the difference $f(x+\omega) - f(x)$ can be made smaller than any given quantity provided w can be taken as small as we please". [Bolzano, 1955, 174-175]. "The true statement is that the continuous function never reaches its top value without first passing through all downs, that is to say that $f(x+n\Delta x)$ can take on any value lying between $f(x)$ and $f(x + \Delta x)$ if n is defined arbitrarily between 0 and 1. However, this statement may not be deemed to be an *explanation* of the notion of continuity, it constitutes a *theorem* of continuity [Bolzano, 1955, italics by Bolzano, p. 175]. "Therefore, this theorem can be stated as follows: If a variable which depends on any other variable x turns to be positive for $x = \alpha$ and negative for $x = \beta$, there is always a value x lying between α and β for which it becomes zero or a value for which it becomes continuous." [Bolzano, 1955, p.176]. This is a theorem, said Bolzano, and it must be proved. Bolzano also noted that such point x need not be unique. He believed it to underlie the algebraic theorem on factorization of a polynomial and Lagrange's theorem on positivity of a definite integral for positive function which equals zero only at the end point of the interval.

Bolzano suggested his own more stringent plan of proof of this theorem based on another more general one: "If two functions x; $f(x)$ and $\phi(x)$ either for all values of x or for all values lying between α and β change in accordance with the law of continuity; if further $f(\alpha) < \phi(\alpha)$ and $f(\beta) > \phi(\beta)$, each time there is value x lying between α and β for which $f(x) = \phi(x)$" [Bolzano, 1955, p. 170-204, 198]. Bolzano proved this theorem assuming that there exists an upper bound of an area where an abstract property of the function[16] is met and using the interval bisecting method in the auxiliary theorem. He demonstrated that the existence of the least upper bound does not cause any inconsistency, as it became possible to provide a stronger demonstration of the existence of a bound only after Weierstrass's works in 1860's and after 1872 when the theory of a real number was developed by Ch. Méray,

[15] The wording of this law belongs to Leibnitz.

[16] e.g. negativeness.

K. Weierstrass, E. Heine, R. Dedekind, and G. Cantor. Bolzano tried to develop the theory of real numbers through section later, in 1830s [Rychlik, 1958].

Thereafter, Bolzano proved the theorem on the root interval. Here, Bolzano stated another theorem: "Passing from one value to another, at least once, the function takes the value of each intermediate value"[17]. Bolzano emphasized that the above property was a result of continuity, however, it could not be taken as basis for defining the continuity.

I would also note that this work contains the criterion for convergence of a sequence [Bolzano, 1955, 188 - 189] which was stated 4 years later by A. Cauchy and is named after him.

1821, Cauchy, the Theorem on the Root Interval in the *Cours d'analyse de l'École royale polytechnique*

In 1821, A.L. Cauchy (1789-1857) published *Cours d' analyse* [Cauchy, 1821] based on his lectures at École Polytechnique. The first part of the course was entitled *Analyse algébrique* and the second part, *Le Calcul infinitésimal,* was published in 1823.

The notion of a continuous function introduced in *Algebraic Analysis* is exactly the same as Bolzano's definition [Cauchy, 1821, p. 43].

Cauchy did not mention Rolle, although he addressed the approximate solution of algebraic equations. He provided the following theorem in the chapter devoted to solution of equations [Cauchy, 1821, p. 378]: "Let $f(x)$ be a real function of the variable x, which remains continuous with respect to this variable between the limits x = x_0, $x = X$. If the two quantities and have opposite signs, we can satisfy the equation (1) $f(x) = 0$ with one or several real values of x contained between $x = x_0$ and $x = X$" [Cauchy, 2009, p. 330]. Cauchy proved this theorem using interval bisection, but, unlike Bolzano, he did not use the notion of the upper bound. Instead, he based his argument on convergent sequences.

What is very important for analysis, is that Cauchy stated the theorem on the intermediate value as a property of a continuous function: "Theorem on a continuous

[17] e.g., continuum

function. If $f(x)$ is a continuous function with respect to the variable x between the limits x = x_0 and $x = X$, and if b denotes a quantity between $f(x_0)$ and $f(X)$, we may always satisfy the equation $f(x) = b$ by one or more real values of x contained between x_0 and X" [Cauchy, 1821, p. 50; Cauchy 2009, p.32].

Cauchy described a couple of methods to solve algebraic equations including the method of Descartes; he compared the methods of Newton and Lagrange through an example of a cubic equation; however, he mentioned nothing about separating the roots of an equation by the roots of its derivative.

1834, Rolle's Theorem in Drobisch

In 1834, M. W. Drobisch (1802-1896), Professor at Leipzig University, published *The Lectures on Equations of the Highest Orders* where he described Rolle's method of cascades, in §107 [Drobisch, 1834., p. 161]. He complained about Rolle's complicated language, but called his method worthy of respect, reasoned and stated it as follows: "These theorems (rules) were obtained incomplete in Rolle's draft. Moreover, the argument is based on the method of cascades which was extremely hard to understand. There was an essential kernel there that to solve the initial equation, one after another, they formed auxiliary equations, which is much like constructing a house using this method. Thus, we successively obtain roots of lower order equations which provide us with reliable limits of the roots of higher order equations which we calculate approximately and will describe later up to the roots of the initial equation. This method is based on the assumption that the initial equation has roots in general. This brings about its limitation and impractical awkwardness instead of search for a straighter way." [Drobisch, 1834., p. 186-188].

Drobisch quoted the theorem on the root interval from Cauchy's course and his mean value theorem in Chapter Seven of *The Alternative Method of Identifying Real and Imaginary Roots* [Drobisch, 1834, p. 161-176]. He stated the theorem as follows: "Two neighboring real roots are divided by the root of the derivative of an equation, the roots of which derivative are, in turn, divided by roots of the next derivative." [Drobisch, 1834, p. 176].

"*Theorem* 1. There is at least one real root of a derivative lying between the two neighboring real roots of the initial equation; however, there may also be 3, 5, or any other odd number of roots between them. *Theorem* 2. There lies no more than one root of the initial equation between the two neighboring real roots of the derivative equation, however, it may also happen that there are no roots at all between them. *Theorem* 3. No more than one real root of the initial equation is larger than the largest real root of the derivative equation; no more than one real root of the initial equation is less than the least root of the derivative equation; however, it may happen that there is no real root of the initial equation which is larger than the largest root and less than the least root of the derivative equation. In this last conclusion, we joined two halves of the 2nd conclusion, i.e. the largest root of the initial equation may lie between the first and the second, or between the second and the third, the third and the fourth, etc. root of the derivative equation[18]″ [Drobisch, 1834, p. 178-179].

The algebraic aspect of the theorem of Rolle in solving equations attracted attention of the Italian mathematician G. Bellavitis (1803-1880) who described the method of Rolle in his book entitled *A simple way to find real roots of algebraic equations and a new method of determination of imaginary roots* [Bellavitis, 1846].

1861, Rolle's theorem in Weierstrass

The concept of continuous functions drastically changed in mid 19th century when new mathematical objects appeared, when it was necessary to classify points of discontinuity and assess the scope of this notion and the possibility of neglecting them when expanding a function in Fourier series. The definition of a continuous function in the language of ″$\varepsilon - \delta$″ was introduced by Karl Weierstrass (1815-1897) in 1861 [27]; E. Heine (1821-1881), R. Dedekind (1831-1916) and G. Cantor (1845-1918) continued developing the concept of continuity in their works in 1870s.

In the summer term (May-June) of 1861, Weierstrass gave a course of lectures on differential calculus at the Königlichen Gewerbeinstitut, in Berlin. Herman Schwarz saved his notes of these lectures [Weierstrass, 1861] which were later published by Pierre Dugac [Dugac, 1972].

[18] Stated in accordance with Lagrange's Résolution de l'équat. Numér. Not. VIII, who might have probably been the first to make any of the above assertions based on the reputed rule of Rolle. *Note of W. Drobisch.*

Weierstrass defined continuous functions and described their properties: "if $f(x)$ is function of x and x is a defined value, then, as x passes to $x + h$, the function will change and will be $f(x + h)$; the difference $f(x + h) - f(x)$ is called the change occurring in the function because of the argument passing from x to $x + h$. If a bound δ can be determined for h such that for all values of h (the absolute value whereof is) less than δ, $f(x + h) - f(x)$ is less than any arbitrary small value ε, then they say that infinitely small changes in the function correspond to the infinitely small changes in the argument. Because they say that any value can become infinitely small, if its absolute value can become less than any arbitrary small value. If a function is such that infinitely small changes in the function correspond to the infinitely small changes in the argument, they say that this is a continuous function of the argument or that it changes continuously with its argument [Yushkevich, 1977, p. 189].

There is a theorem as follows, under the heading Study of *changes in functions:* "If $f(x_1) = f(x_2)$ for two defined values of x_1 and x_2 of the argument, then there must be at least one value of x_0 between x_1 and x_2 for which the first derivative $f'(x_0)$ equals zero." A.P. Yushkevich believes, "this is the first or one of the first statements of the so-called theorem of Rolle." [Yushkevich, 1977, p. 193].

Later, in 1886, analyzing the extension of the notion of a function, Weierstrass wrote that earlier only functions represented by rational number expression, e.g. those with rational coefficients, were considered. "They changed in accordance with the law of continuity, and this was all we knew about the function. But the discovery of Fourier's series has shown that this is not true; there are continuous functions which may not be obtained by representation as before. There can always be found a mathematical expression for a strictly determined continuous function. Therefore, properties of any function can be derived from the basic notions of continuity, as it is important for any research to derive further notions from the basic ones" [Weierstrass, 1989, p. 21].

The theorem on the root interval, mean value theorem, and theorem on the root of a derivative obtained the status of properties of continuous functions. There were eleven properties like that stated in Dini's works, while there were about four such properties formulated during the time of Cauchy.

1878, Rolle's Theorem in Dini

In 1878, a *Course of Lectures on the Theory of Functions of a Real Variable* by U. Dini [Dini, 1878] (1845-1918), Professor at Pisa University, was published. It was for the first time that the definition of a continuous function was introduced in this course through unilateral limits. The theorem of Rolle was anonymously stated as follows: "If function $f(x)$ in interval (α, β) is finite and continuous at all points but ends of the interval and has a finite and specified or an infinite and specified derivative, and in addition, has the same values at the extreme points a and b, then there is at least one point x' in interval (a, b) for which point $f'(x') = 0$" [Dini, 1878, p. 76-77]. (a, b in (α, β)).

1879, Rolle's Theorem in Cantor

During the period from 1874 to 1884, G. Cantor (1845-1918) wrote his main articles on set theory. In 1872, he began to create his theory of real numbers; in 1874, he proved countability of the set of algebraic numbers; in 1878, he developed the notion of the power set and considered the problem of comparing the power sets of continuous manifolds of any different dimensions, and came to a paradoxical conclusion that all of them have the same power set and are equivalent to that of a unit segment. "I can see but cannot believe it", wrote Cantor to Dedekind. Cantor concluded that the notion of dimensionality must be based on mutually continuous mapping of varieties on one another.

In 1879, Cantor tried to prove the theorem that two continuous varieties M_μ and M_ν with dimensions μ and ν, where $\mu < \nu$, cannot be mapped one-to-one onto one another continuously. To prove this theorem, Cantor used the theorem of Rolle on the root interval. The remarkable fact is that Cantor referred to Cauchy's Course of Analysis of 1821. However, the scheme of his verbal proof was very similar to Bolzano's interpretation in 1817. This attempt to prove the theorem was stated by Cantor in his article entitled *Über einen Satz aus der Theorie der stetigen Mannigfaltigkeiten* (On a theorem from the theory of continuous manifolds) [Cantor, 1985, p. 36-39]. His proof was not complete[19]. However, the theorem on the root

[19] It was L.E.Y. Brauer who provided the first satisfactory proof of the general theorem that the varieties of different dimensions cannot be mapped one-to-one onto one another at the same time mutually continuously, in *Math. Ann.* of 1910, Bd. 70, s. 161-165.

interval took a fundamental meaning in his works for the notions of continuity and dimensionality.

1886, Weierstrass and substantiation of continuity

In the summer term of 1886 (May-June), Weierstrass lectured on substantiation of the theory of analytic functions. Based on the notion of a limiting point Weierstrass developed the notion of the least upper bound[20] using the theorem on the root interval. Based on this theorem he introduced the notion of connectivity: "proceeding from any point of the continuum, we will always remain there" [Weierstrass, 1989, p. 70].

Many mathematicians of the 19th century addressed these theorems; in many cases, their analysis was pretty interesting. Unfortunately, the works of Herman Hankel and many other mathematicians, in this respect, are beyond the scope of this article.

Conclusion

Analysis of algebraic equations has led to two fundamental statements in the theory of functions (the theorem on the root interval and theorem of the root of a derivative) being formulated and the theorem of mean value being created. In Russian sources, various authors call the theorem on the root interval either *the second Rolle's theorem* (Shatunovsky [Shatunovsky, 1923, p. 121-122]), or the *theorem of Bolzano Cauchy* (Shatunovsky's student G.M. Fihtengolc [Fihtengolc, p. 128]). The mean value theorem in Russia is called the second theorem of Bolzano-Cauchy [Fichtengolts, p. 131].

It took three hundred years for one of the most fundamental theorems in analysis to take its final shape. This theorem is not only of great importance in terms of methodology, but has great applied significance as well, e.g. in differential geometry, functional analysis, mechanics. N. N. Luzin (1883-1950) proved the theorem on osculating circle and theorem on the center of curvature [Luzin, 1946,

[20] In doing so Weierstrass used variational methods.

p. 317] with the help of this theorem. Initially intended for polynomials, Rolle's theorem was extended to continuous functions and enriched their properties. Luzin said that "this theorem underlies the theoretical development of differential and integral calculus" [Luzin, 1946, p. 317].

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